Precise rates in the law of the logarithm in the Hilbert space*

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Abstract. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables taking values in a real separable Hilbert space $(H, \|\cdot\|)$ with covariance operator Σ, and set $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Let $a_n = o(\sqrt{n/\log n})$. We prove that, for any $1 < r < 3/2$ and $a > -d/2$,

$$\begin{aligned}&\lim_{\varepsilon \searrow \sqrt{r-1}} \left[\varepsilon^2 - (r-1)\right]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} (\log n)^a \mathsf{P}\Big\{\|S_n\| \geq \sigma\phi(n)\varepsilon + a_n\Big\} \\ &= \Gamma^{-1}(d/2) K(\Sigma)(r-1)^{\frac{d-2}{2}} \Gamma(a + d/2)\end{aligned}$$

holds if

$$\mathsf{E}X = 0, \quad \mathsf{E}[(X, y)]^2 < \infty, \quad \mathsf{E}\left[\|X\|^{2r} (\log \|X\|)^{a-r}\right] < \infty,$$

and

$$\mathsf{E}[(X, e_i)^2 I\{|(X, e_i)| > t\}] = o(\frac{1}{\log t}), \qquad \text{as} \quad t \to \infty, \quad \text{for} \quad \forall i.$$

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1 Introduction and main results.

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent identically distributed random variables (r.v.'s) and set $S_n = \sum_{k=1}^n X_k$, $M_n = \max_{k\leq n} |S_k|$, for $n \geq 1$. Also let $\log x = \ln(x \vee e)$, $\log\log x = \log(\log x)$ and $\phi(x) = \sqrt{2x \log x}$. The following is the well known complete convergence firstly established by Hsu and Robbins (1947):

$$\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \geq \varepsilon n\} < \infty, \quad \varepsilon > 0$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$. Baum and Katz (1965) extended this result and proved the following theorem.

Theorem A *Let $1 \leq p < 2$ and $r \geq p$. Then*

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\{|S_n| \geq \varepsilon n^{1/p}\} < \infty, \quad \varepsilon > 0$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}|X|^{rp} < \infty$.

Many authors considered various extensions of the results of Hsu-Robbins and Baum-Katz. Some of them studied the precise asymptotics of the infinite sums as $\varepsilon \to 0$ (c.f. Heyde (1975), Chen (1978), Spătaru (1999) and Gut and Spătaru (2000a)). But, this kind of results do not hold for $p = 2$. However, by replacing $n^{1/p}$ by $\sqrt{n \log\log n}$, Gut and Spătaru (2000b) established an analogous result called the precise asymptotics of the law of the iterated logarithm, and Zhang (2001) gave the sufficient and necessary conditions for such kind of results to hold. By replacing $n^{1/p}$ by $\sqrt{n \log n}$, Lai (1974) and Chow and Lai (1975) considered the following result on the law of the logarithm.

Theorem B *Suppose that $\mathsf{Var}X = \sigma^2$ and $r \geq 1$. Then the following are equivalent:*

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\{M_n \geq \varepsilon \phi(n)\} < \infty, \quad \text{for all } \ \varepsilon > \sigma\sqrt{r-1};$$

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\{|S_n| \geq \varepsilon \phi(n)\} < \infty, \quad \text{for all } \ \varepsilon > \sigma\sqrt{r-1};$$

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\{|S_n| \geq \varepsilon \phi(n)\} < \infty, \quad \text{for some } \ \varepsilon > 0;$$

$$\mathsf{E}X = 0 \ \text{ and } \ \mathsf{E}|X|^{2r}/(\log|X|)^r < \infty.$$

For $r = 1$, Gut and Spătaru (2000a) gave the following precise asymptotics.

Theorem C *Suppose that $EX = 0$ and $EX^2 = \sigma^2 < \infty$. Then, for $0 \le \delta \le 1$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\delta+2} \sum_{n=1}^{\infty} n^{-1} (\log n)^{\delta} P\{|S_n| \ge \varepsilon \sqrt{n \log n}\} = \frac{\mu^{(2\delta+2)}}{\delta + 1} \sigma^{2\delta+2},$$

where $\mu^{(2\delta+2)}$ is the $(2\delta+2)$th absolute moment of the standard normal distribution.

Recently Zhang (2003) gave the precise asymptotics for all $r > 1$ and obtained the sufficient and necessary conditions for such kind of results to hold. The purpose of this paper is to find out whether there are the analogues in the Hilbert space setting.

In the context, let $\{X, X_n; n \ge 1\}$ be a sequence of independent identically distributed random variables (r.v.'s) taking values in a real separable Hilbert space $(\mathbf{H}, \|\cdot\|)$ with mean zero and covariance operator Σ. Denote the largest eigenvalue of Σ by σ^2, i.e. $\sigma^2 := \sup\{\mathsf{E}[(X, y)^2] : \|y\| \le 1\}$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{H}$. And let d be the dimension of the corresponding eigenspace. Let σ_i^2, $1 \le i < d'$ be the positive eigenvalues of Σ arranged in a nonincreasing order and taking into account the multiplicities. Further, if $d' < \infty$, put $\sigma_i^2 = 0$, $i \ge d'$. Note that we always have $\sigma_i^2 = \sigma^2$, $1 \le i \le d$ and $\sigma_i^2 < \sigma^2$, $i > d$. Write $\{e_i\}$ be a sequence of orthonormal eigenvectors corresponding to the eigenvalues $\{\sigma_i^2\}$. Set $S_n = \sum_{k=1}^n X_k$, $n \ge 1$. The following theorems are our main results.

Theorem 1.1 *Let $1 < r < 3/2$ and $a > -d/2$ and let $a_n(\varepsilon)$ be a function of ε such that*

$$a_n(\varepsilon) \log n \to \tau, \quad \text{as } n \to \infty \text{ and } \varepsilon \searrow \sqrt{r-1}. \tag{1.1}$$

Suppose $\{f_n\}$ is a sequence of non-negative numbers satisfying

$$F_n := \sum_{k=1}^n f_k \sim \sum_{k=1}^n (\log k)^a, \quad n \to \infty. \tag{1.2}$$

Assume

$$EX = 0,\ E[(X, y)]^2 < \infty, \quad \forall y \in \mathbf{H}, \tag{1.3}$$

$$E\big[\|X\|^{2r} (\log \|X\|)^{a-r}\big] < \infty \tag{1.4}$$

and

$$E\left[(X, e_i)^2 I\{|(X, e_i)| > t\}\right] = o\Big(\frac{1}{\log t}\Big), \quad \text{as } t \to \infty, \quad \forall i. \tag{1.5}$$

Then

$$\begin{aligned} &\lim_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} f_n P\Big\{\|S_n\| \ge \sigma \phi(n) \big(\varepsilon + a_n(\varepsilon)\big)\Big\} \\ &= \Gamma^{-1}(d/2) K(\Sigma) (r-1)^{\frac{d-2}{2}} \Gamma(a + d/2) \exp\{-2\tau\sqrt{r-1}\}, \end{aligned} \tag{1.6}$$

where $\Gamma(\cdot)$ *is a gamma function and* $K(\Sigma) := \prod_{i=d+1}^{\infty}(1-\sigma_i^2/\sigma^2)^{-1/2}$.

Letting $f_n = 1$ and $\tau = 0$ yields the following corollary.

Corollary 1.1 *Let* $1 < r < 3/2$ *and* $a_n = o(\sqrt{n/\log n})$. *Suppose (1.3), (1.4) and (1.5). Then*

$$\lim_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{d/2} \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\Big\{\|S_n\| \geq \varepsilon\sigma\phi(n) + a_n\Big\} = K(\Sigma)(r-1)^{\frac{d-2}{2}}. \tag{1.7}$$

Conjecture We believe that Theorem 1.1 and Corollary 1.1 hold as well for $r \geq 3/2$. To get such an improvement of the results, we think a different approach is necessary.

Theorem 1.2 *Let* $a > -1$ *and* $a_n = O(1/\log n)$. *Suppose* $\{f_n\}$ *is a sequence of non-negative numbers satisfying (1.2). Assume (1.3) and*

$$\mathsf{E}\big[\|X\|^2 (\log \|X\|)^{a+4}\big] < \infty. \tag{1.8}$$

Then

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(a+1)} \sum_{n=1}^{\infty} n^{-1} f_n \mathsf{P}\big\{\|S_n\| \geq \sigma\phi(n)(\varepsilon + a_n)\big\} = (2\sigma^2)^{-(a+1)}(a+1)^{-1}\mathsf{E}[\|Y\|]^{2(a+1)}, \tag{1.9}$$

where Y *is a Gaussian r.v. taking value in a real separable Hilbert space with mean zero and covariance operator* Σ.

The proofs consist of two stages. Firstly we verify the theorems under the assumption that X is a nondegenerate Gaussian random variable with mean zero and covariance operator Σ in Section 2, after which, by using the truncation and approximation method, we then show the general cases. Throughout this paper, we let $K(\alpha, \beta, \cdots)$, $C(\alpha, \beta, \cdots)$ etc. denote positive constants which depend on $\alpha, \beta, \cdots$ only, whose values can differ in different places. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$, as $n \to \infty$, and $a_n \approx b_n$ means that $C_0^{-1} b_n \leq a_n \leq C_0 b_n$ for some $C_0 > 0$ and all n large enough.

2 Normal cases.

In this section, we prove Theorem 1.1 and Theorem 1.2 in the case that $\{X, X_n; n \geq 1\}$ are Gaussian random variables. Let Y be a nondegenerate Gaussian mean zero r.v. with covariance operator Σ, say. Denote the density of $\|Y\|^2$ by g. Our results are as follows.

Proposition 2.1 *Let* $r > 1$ *and* $a > -d/2$. *Suppose* $a_n(\varepsilon)$ *is a function of* ε *satisfying (1.1) and* $\{f_n\}$ *is a sequence of non-negative numbers satisfying (1.2). Then*

$$\lim_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|Y\| \geq \sigma\sqrt{2\log n}\big(\varepsilon + a_n(\varepsilon)\big)\Big\}$$
$$= \Gamma^{-1}(d/2)K(\Sigma)(r-1)^{\frac{d-2}{2}}\Gamma(a+d/2)\exp\{-2\tau\sqrt{r-1}\}, \tag{2.1}$$

where Γ and $K(\Sigma)$ are as in Theorem 1.1.

Proposition 2.2 *Let $a > -1$ and $a_n = O(1/\log n)$. Suppose $\{f_n\}$ is a sequence of non-negative numbers satisfying (1.2). Then we have*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2(a+1)} \sum_{n=1}^{\infty} n^{-1} f_n P\big\{\|Y\| \geq \sigma(\varepsilon + a_n)\sqrt{2\log n}\big\} = (2\sigma^2)^{-(a+1)}(a+1)^{-1} E[\|Y\|]^{2(a+1)}. \tag{2.2}$$

The following lemmas will be used in the proofs of the propositions.

Lemma 2.1 *Let Y be a nondegenerate Gaussian mean zero r.v. with covariance operator Σ. Then for $y > 0$,*

$$P\{\|Y\| > y\} \sim 2A\sigma^2 y^{d-2} \exp\{-y^2/(2\sigma^2)\}, \qquad \text{as} \quad y \to \infty, \tag{2.3}$$

where $A := (2\sigma^2)^{-d/2}\Gamma^{-1}(d/2)K(\Sigma)$.

Proof. Note the result of Zolotarev (1961) that

$$\lim_{y\to\infty} \Big\{ g(y)/\big(y^{d/2-1}\exp\{-y/(2\sigma^2)\}\big)\Big\} = A, \tag{2.4}$$

we can get the result immediately.

Lemma 2.2 *For any $\rho_0 > 0$, there exists a constant $C_{\rho_0} = C(\rho_0) > 0$ such that*

$$g(y) \leq C_{\rho_0} y^{d/2-1}\exp\{-y/(2\sigma^2)\}, \quad \text{for all} \;\; y \in [\rho_0, +\infty). \tag{2.5}$$

Proof. Notice that $g(y)/\big(y^{d/2-1}\exp\{-y/(2\sigma^2)\}\big)$ is continuous on $[\rho_0, +\infty)$. By (2.4) the result follows.

Lemma 2.3 *Let $a_n > 0$, $b_n > 0$, $c_n > 0$, $A_n = \sum_{k=1}^n a_k$, $B_n = \sum_{k=1}^n b_k$. Suppose that*

$$A_n \sim B_n \;\; \text{and} \;\; \sum_{k=1}^n b_k c_k \to \infty, \qquad \text{as } n \to \infty.$$

Further, suppose one of the following conditions is satisfied:

(i) *The sequence $\{c_n\}$ is eventually non-increasing;*

(ii) *The sequence $\{c_n\}$ is eventually non-decreasing, and*

$$\sum_{k=1}^n b_k c_k \approx B_n c_{n+1}. \tag{2.6}$$

Then we have

$$\sum_{k=1}^n a_k c_k \sim \sum_{k=1}^n b_k c_k.$$

Proof. We only show the result under the condition (ii). At that case, for any $\theta > 1$, there exists a $n_0 \geq 1$ such that for all $n \geq n_0$, c_n is non-decreasing and $\theta^{-1}A_n \leq B_n \leq \theta A_n$. Then by the Abel transform, for $\forall n > n_0$

$$\begin{aligned}
& \sum_{k=1}^{n} a_k c_k = \sum_{k=1}^{n} A_k(c_k - c_{k+1}) + A_n c_{n+1} \\
\leq & \sum_{k=1}^{n_0} A_k(c_k - c_{k+1}) + \sum_{k=n_0+1}^{n} \theta^{-1} B_k(c_k - c_{k+1}) + \theta B_n c_{n+1} \\
= & \sum_{k=1}^{n_0} (A_k - \theta^{-1}B_k)(c_k - c_{k+1}) + \sum_{k=1}^{n} \theta^{-1} B_k(c_k - c_{k+1}) + \theta^{-1} B_n c_{n+1} + (\theta - \theta^{-1}) B_n c_{n+1} \\
= & \sum_{k=1}^{n_0} (A_k - \theta^{-1}B_k)(c_k - c_{k+1}) + \theta^{-1} \sum_{k=1}^{n} b_k c_k + (\theta - \theta^{-1}) B_n c_{n+1}.
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n a_k c_k}{\sum_{k=1}^n b_k c_k} \leq \theta^{-1} + (\theta - \theta^{-1})K.$$

Letting $\theta \to 1$, we get

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n a_k c_k}{\sum_{k=1}^n b_k c_k} \leq 1.$$

Similarly,

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n a_k c_k}{\sum_{k=1}^n b_k c_k} \geq 1.$$

The proof is completed.

Lemma 2.4 *For $n \geq 1$, let $\alpha_n(\varepsilon) > 0$, $\beta_n(\varepsilon) > 0$ and $f(\varepsilon) > 0$ satisfying*

$$\alpha_n(\varepsilon) \sim \beta_n(\varepsilon), \quad \text{as } n \to \infty \text{ and } \varepsilon \to \varepsilon_0,$$

and,

$$f(\varepsilon)\beta_n(\varepsilon) \to 0, \quad \text{as } \varepsilon \to \varepsilon_0, \quad \forall n \geq 1.$$

Then

$$\limsup_{\varepsilon\to\varepsilon_0} (\liminf_{\varepsilon\to\varepsilon_0}) f(\varepsilon) \sum_{n=1}^{\infty} \alpha_n(\varepsilon) = \limsup_{\varepsilon\to\varepsilon_0} (\liminf_{\varepsilon\to\varepsilon_0}) f(\varepsilon) \sum_{n=1}^{\infty} \beta_n(\varepsilon).$$

Proof. For any $\theta > 1$, there exist $n_0 \geq 1$ and a neighborhood U of ε_0 such that

$$\theta^{-1}\beta_n(\varepsilon) \leq \alpha_n(\varepsilon) \leq \theta\beta_n(\varepsilon), \qquad \text{for } n \geq n_0, \quad \varepsilon \in U.$$

Then

$$\theta^{-1} \sum_{n=n_0}^{\infty} \beta_n(\varepsilon) \leq \sum_{n=n_0}^{\infty} \alpha_n(\varepsilon) \leq \theta \sum_{n=n_0}^{\infty} \beta_n(\varepsilon), \qquad \text{for } \varepsilon \in U.$$

Now, the result follows easily.

Lemma 2.5 *Let* $a_n > 0$, $c_n > 0$, *and* $A_n = \sum_{k=1}^n a_k$, *for* $n \geq 1$. *Suppose that the sequence* $\{c_n\}$ *is non-increasing and* $A_n c_n \to 0$, *as* $n \to \infty$. *Then*

$$\sum_{n=1}^{\infty} a_n c_n = \sum_{n=1}^{\infty} A_n (c_n - c_{n+1}).$$

Proof. By the Abel transform, we can get the result immediately.

Lemma 2.6 *Let* $a_n > 0$, $b_n > 0$, $c_n > 0$, $A_n = \sum_{k=1}^n a_k$ *and* $B_n = \sum_{k=1}^n b_k$, *for* $n \geq 1$. *Suppose that the sequence* $\{c_n\}$ *is non-increasing, and* $A_n \leq B_n$, $\forall n \geq 1$. *Then*

$$\sum_{k=1}^{\infty} a_k c_k \leq \sum_{k=1}^{\infty} b_k c_k \quad \text{and} \quad \sum_{k=j}^{\infty} a_k c_k \leq \sum_{k=j}^{\infty} b_k c_k + B_{j-1} c_j,$$

for any $j \geq 1$, *where* $B_0 = 0$.

Proof. From the Abel transform, it follows that

$$\begin{aligned}
\sum_{k=1}^{n} a_k c_k &= \sum_{k=1}^{n} A_k (c_k - c_{k+1}) + A_n c_{n+1} \\
&\leq \sum_{k=1}^{n} B_k (c_k - c_{k+1}) + B_n c_{n+1} = \sum_{k=1}^{n} b_k c_k
\end{aligned}$$

and for any $j \geq 1$,

$$\begin{aligned}
\sum_{k=j}^{n} a_k c_k &= \sum_{k=j}^{n} A_k (c_k - c_{k+1}) + A_n c_{n+1} - A_{j-1} c_j \\
&\leq \sum_{k=j}^{n} B_k (c_k - c_{k+1}) + B_n c_{n+1} = \sum_{k=j}^{n} b_k c_k + B_{j-1} c_j.
\end{aligned}$$

The results follow.

Now, we turn to prove the propositions.

Proof Proposition 2.1. Firstly, note that the limit in (2.1) does not depend on any finite terms of the infinite series. Secondly, by Lemma 2.1 and the condition (1.1), we have

$$\begin{aligned}
&\mathsf{P}\Big\{\|Y\| \geq \sigma\sqrt{2\log n}\big(\varepsilon + a_n(\varepsilon)\big)\Big\} \\
&\quad \sim 2A\sigma^2 \left(\sigma\big(\varepsilon + a_n(\varepsilon)\big)\sqrt{2\log n}\right)^{d-2} \exp\{-\big(\varepsilon + a_n(\varepsilon)\big)^2 \log n\} \\
&\quad \sim 2A\sigma^d \big(\varepsilon\sqrt{2\log n}\big)^{d-2} \exp\{-\varepsilon^2 \log n\} \exp\{-2\varepsilon a_n(\varepsilon) \log n\} \\
&\quad \sim 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}} (\log n)^{\frac{d-2}{2}} \exp\{-\varepsilon^2 \log n\} \exp\{-2\tau\sqrt{r-1}\},
\end{aligned} \tag{2.7}$$

as $n \to \infty$, $\varepsilon \searrow \sqrt{r-1}$, where A is as in Lemma 2.1. Also, by (1.2) and Lemma 2.3, we have

$$A_n := \sum_{k=1}^{n} k^{r-2} (\log k)^{\frac{d-2}{2}} f_k \sim B_n := \sum_{k=1}^{n} k^{r-2} (\log k)^{a-1+d/2} \approx n^{r-1} (\log n)^{a-1+d/2}. \tag{2.8}$$

Then we conclude that

$$
\begin{aligned}
& \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|Y\|\geq \sigma\sqrt{2\log n}(\varepsilon+a_n(\varepsilon))\Big\} \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty} n^{r-2} f_n 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}(\log n)^{\frac{d-2}{2}}\exp\{-\varepsilon^2\log n\}\exp\{-2\tau\sqrt{r-1}\} \\
& \qquad\qquad (\text{ by (2.7) and Lemma 2.4}) \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty} A_n\Big\{\exp\{-\varepsilon^2\log n\}-\exp\{-\varepsilon^2\log(n+1)\}\Big\} \\
& \qquad\qquad \cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \qquad\qquad (\text{ by (2.8) and Lemma 2.5}) \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty} B_n\Big\{\exp\{-\varepsilon^2\log n\}-\exp\{-\varepsilon^2\log(n+1)\}\Big\} \\
& \qquad\qquad \cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \qquad\qquad (\text{ by (2.8) and Lemma 2.4}) \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty} n^{r-2}(\log n)^{a-1+d/2}\exp\Big\{-\varepsilon^2\log n\Big\} \\
& \qquad\qquad \cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \qquad\qquad (\text{ by (2.8) and Lemma 2.5}) \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty}\int_n^{n+1} x^{r-2}(\log x)^{a-1+d/2}\exp\Big\{-\varepsilon^2\log x\Big\}dx \\
& \qquad\qquad \cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \qquad\qquad (\text{ by Lemma 2.4}) \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\int_e^{\infty} x^{r-2}(\log x)^{a-1+d/2}\exp\Big\{-\varepsilon^2\log x\Big\}dx \\
& \qquad\qquad \cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\int_1^{\infty} y^{a-1+d/2}\exp\Big\{-\big(\varepsilon^2-(r-1)\big)y\Big\}dy \\
& \qquad\qquad \cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \\
= \; & \limsup_{\varepsilon\searrow\sqrt{r-1}} \int_{\varepsilon^2-(r-1)}^{\infty} z^{a-1+d/2}\exp\{-z\}dz\cdot 2^{\frac{d}{2}} A\sigma^d (r-1)^{\frac{d-2}{2}}\exp\{-2\tau\sqrt{r-1}\} \\
= \; & \Gamma^{-1}(d/2)K(\Sigma)(r-1)^{\frac{d-2}{2}}\Gamma(a+d/2)\exp\{-2\tau\sqrt{r-1}\}.
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
& \liminf_{\varepsilon\searrow\sqrt{r-1}} [\varepsilon^2-(r-1)]^{a+d/2}\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|Y\|\geq \sigma\sqrt{2\log n}(\varepsilon+a_n(\varepsilon))\Big\} \\
= \; & \Gamma^{-1}(d/2)K(\Sigma)(r-1)^{\frac{d-2}{2}}\Gamma(a+d/2)\exp\{-2\tau\sqrt{r-1}\}.
\end{aligned}
$$

Then (2.1) is proved.

Proof Proposition 2.2. Without losing of generality, we can assume that $|a_n|\leq \tau_0/\log n$, $\tau_0>0$. Fix $0<\delta<1$. For any $0<\varepsilon<\delta/2$, if $\varepsilon^2>\delta\tau_0/\log n$, then

$$
2\sigma^2(\varepsilon-\tau_0/\log n)^2\log n \geq 2\sigma^2\delta\tau_0\big(1-\sqrt{\frac{\tau_0}{\delta\log n}}\big)^2\geq \sigma^2\delta\tau_0/2>0.
$$

Then by Lemma 2.2, for any $0 < \varepsilon < \delta/2$ small enough and n with $\varepsilon^2 > \delta\tau_0/\log n$,

$$\begin{aligned}
&\zeta_n := \Big|\mathsf{P}\Big\{\|Y\| \ge \sigma\varepsilon\sqrt{2\log n}\Big\} - \mathsf{P}\Big\{\|Y\| \ge \sigma(\varepsilon + a_n)\sqrt{2\log n}\Big\}\Big| \\
\le\ & 2\Big|\mathsf{P}\Big\{\|Y\| \ge \sigma\big(\varepsilon + \frac{\tau_0}{\log n}\big)\sqrt{2\log n}\Big\} - \mathsf{P}\Big\{\|Y\| \ge \sigma\big(\varepsilon - \frac{\tau_0}{\log n}\big)\sqrt{2\log n}\Big\}\Big| \\
\le\ & 2\int_{2\sigma^2(\varepsilon-\tau_0/\log n)^2\log n}^{2\sigma^2(\varepsilon+\tau_0/\log n)^2\log n} g(z)dz \\
\le\ & C\int_{2\sigma^2(\varepsilon-\tau_0/\log n)^2\log n}^{2\sigma^2(\varepsilon+\tau_0/\log n)^2\log n} z^{d/2-1}\exp\{-z/(2\sigma^2)\}dz \\
\le\ & C\exp\{-\varepsilon^2\log n\}\int_{2\sigma^2(\varepsilon-\tau_0/\log n)^2\log n}^{2\sigma^2(\varepsilon+\tau_0/\log n)^2\log n} z^{d/2-1}dz.
\end{aligned}$$

Noting that for $d \ge 2$,

$$\begin{aligned}
&\zeta_n \le C\exp\{-\varepsilon^2\log n\}\big\{2\sigma^2\big(\varepsilon + \frac{\tau_0}{\log n}\big)^2\log n\big\}^{d/2-1}\cdot 4\sigma^2\tau_0\varepsilon \\
\le\ & C\exp\{-\varepsilon^2\log n\}\big\{\big(\varepsilon + \frac{\varepsilon}{2}\big)^2\log n\big\}^{d/2-1}\cdot\varepsilon \\
\le\ & C\varepsilon^{d-1}\exp\{-\varepsilon^2\log n\}(\log n)^{d/2-1},
\end{aligned}$$

and, for $d = 1$,

$$\begin{aligned}
&\zeta_n \le C\exp\{-\varepsilon^2\log n\}\big\{2\sigma^2\big(\varepsilon - \frac{\tau_0}{\log n}\big)^2\log n\big\}^{-1/2}\cdot 4\sigma^2\tau_0\varepsilon \\
\le\ & C\exp\{-\varepsilon^2\log n\}\big\{\big(\varepsilon - \frac{\varepsilon}{2}\big)^2\log n\big\}^{-1/2}\cdot\varepsilon \\
\le\ & C\exp\{-\varepsilon^2\log n\}(\log n)^{-1/2}.
\end{aligned}$$

So we have

$$\zeta_n \le C\varepsilon^{d-1}\exp\{-\varepsilon^2\log n\}(\log n)^{d/2-1}$$

for all $d \ge 1$. Hence, for any $0 < \delta < 1$ and each $a > -1$,

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n:\varepsilon^2>\delta\tau_0/\log n} n^{-1}f_n\zeta_n \\
\le\ & C\lim_{\varepsilon\searrow 0}\varepsilon^{2a+d+1}\sum_{n:\varepsilon^2>\delta\tau_0/\log n} n^{-1}f_n\exp\{-\varepsilon^2\log n\}(\log n)^{d/2-1} \\
\le\ & C\lim_{\varepsilon\searrow 0}\varepsilon^{2a+d+1}\sum_{n:\varepsilon^2>\delta\tau_0/\log n} n^{-1}(\log n)^{a+d/2-1}\exp\{-\varepsilon^2\log n\} \\
&\qquad\text{(by Lemma 2.6)} \\
\le\ & C\lim_{\varepsilon\searrow 0}\varepsilon^{2a+d+1}\int_{\exp\{\delta\tau_0/\varepsilon^2\}}^{\infty}\frac{1}{x}(\log x)^{a+\frac{d}{2}-1}\exp\{-\varepsilon^2\log x\}dx \\
=\ & C\lim_{\varepsilon\searrow 0}\varepsilon\int_{\delta\tau_0}^{\infty} y^{a+\frac{d}{2}-1}\exp\{-y\}dy = 0.
\end{aligned}$$

Also,

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n:\varepsilon^2\le\delta\tau_0/\log n} n^{-1}f_n\zeta_n \le 2\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n:\varepsilon^2\le\delta\tau_0/\log n} n^{-1}f_n \\
\le\ & C\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n:\varepsilon^2\le\delta\tau_0/\log n} n^{-1}(\log n)^a \quad \text{(by Lemma 2.3)} \\
\le\ & C\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\int_e^{\exp\{\delta\tau_0/\varepsilon^2\}}\frac{1}{x}(\log x)^a dx \\
\le\ & C\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\int_1^{\delta\tau_0/\varepsilon^2} y^a dy = \frac{C}{a+1}(\delta\tau_0)^{a+1}\to 0, \qquad \text{as} \quad \delta\to 0.
\end{aligned}
$$

It follows that

$$
\lim_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty} n^{-1}f_n\zeta_n = 0. \tag{2.9}
$$

By Lemma 2.3, we have

$$
A_n' := \sum_{k=1}^{n} k^{-1}f_k \sim B_n' := \sum_{k=1}^{n} k^{-1}(\log k)^a. \tag{2.10}
$$

Hence, for any $a>-1$ and $d\ge 1$,

$$
\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty} n^{-1}f_n\mathsf{P}\Big\{\|Y\|\ge\sigma(\varepsilon+a_n)\sqrt{2\log n}\Big\} \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty} n^{-1}f_n\mathsf{P}\Big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log n}\Big\} \quad \text{(by (2.9))} \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty} A_n'\Big[\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log n}\big\}-\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log(n+1)}\big\}\Big] \\
& \text{(by Lemma 2.5)} \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty} B_n'\Big[\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log n}\big\}-\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log(n+1)}\big\}\Big] \\
& \text{(by (2.10) and Lemma 2.4)} \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty} n^{-1}(\log n)^a\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log n}\big\} \\
& \text{(by Lemma 2.5)} \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\sum_{n=1}^{\infty}\int_n^{n+1} x^{-1}(\log x)^a\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log x}\big\}dx \\
& \text{(by Lemma 2.4)} \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\int_e^{\infty} x^{-1}(\log x)^a\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2\log x}\big\}dx \\
=\ & \limsup_{\varepsilon\searrow 0}\varepsilon^{2(a+1)}\int_1^{\infty} y^a\mathsf{P}\big\{\|Y\|\ge\sigma\varepsilon\sqrt{2y}\big\}dy = 2^{-a}\sigma^{-2a-2}\limsup_{\varepsilon\searrow 0}\int_{\sqrt{2}\sigma\varepsilon}^{\infty} z^{2a+1}\mathsf{P}\big\{\|Y\|\ge z\big\}dz \\
=\ & (2\sigma^2)^{-(a+1)}(a+1)^{-1}\mathsf{E}\big[\|Y\|\big]^{2(a+1)}.
\end{aligned}
$$

Similarly, we can get the result of "$\liminf$". So the proposition is now proved.

3 The general cases.

In this section, we will use Feller's (1945) and Einmahl's (1989) truncation methods to show the general cases. Without losing of generality, we assume that $\sigma = 1$ in the sequel. Let $p > 0$, whose value will be special in the proofs of Theorem 1.1 and Theorem 1.2 respectively. And for each $n \ge 1$ and $1 \le j \le n$, we let

$$X'_{nj} = X_j I\{\|X_j\| \le \sqrt{n}/(\log n)^p\}, \qquad \overline{X}'_{nj} = X'_{nj} - \mathsf{E}[X'_{nj}],$$
$$S'_{nj} = \sum_{i=1}^{j} X'_{ni}, \qquad \overline{S}'_{nj} = \sum_{i=1}^{j} \overline{X}'_{ni}$$

and

$$X''_{nj} = X_j I\{\sqrt{n}/(\log n)^p < \|X_j\| \le \phi(n)\}, \qquad \overline{X}''_{nj} = X''_{nj} - \mathsf{E}[X''_{nj}],$$
$$X'''_{nj} = X_j I\{\|X_j\| > \phi(n)\}, \qquad \overline{X}'''_{nj} = X'''_{nj} - \mathsf{E}[X'''_{nj}].$$

And also define S''_{nj}, S'''_{nj}, $\overline{S}''_{nj}$ and $\overline{S}'''_{nj}$ similarly. It is easily seen that under the condition (1.4),

$$\frac{\mathsf{E}\|\overline{S}'_{nn}\|}{\phi(n)} \to 0, \quad \frac{\mathsf{E}\|\overline{S}''_{nn}\|}{\phi(n)} \to 0 \ \text{ and } \ \frac{\mathsf{E}\|\overline{S}'''_{nn}\|}{\phi(n)} \to 0, \tag{3.1}$$

as $n \to \infty$. In fact, to obtain (3.1), we only need the condition

$$\mathsf{E}\big[\|X\|^2 / \log \|X\|\big] < \infty.$$

The proofs of theorems depend on the following lemmas.

Lemma 3.1 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent mean zero H-valued random variables such that for some $Q > 2$, $\mathsf{E}[\|\xi_j\|^Q] < \infty$, $1 \le j \le n$, $n \ge 1$. Then for any $t > 0$*

$$\mathsf{P}\{\|\sum_{j=1}^{n} \xi_j\| \ge t + 18Q^2 \mathsf{E}\|\sum_{j=1}^{n} \xi_j\|\} \le \exp\{-t^2/(144\Lambda_n)\} + C_1 \sum_{j=1}^{n} \mathsf{E}[\|\xi_j\|^Q]/t^Q, \tag{3.2}$$

where $\Lambda_n := \sup\{\sum_{j=1}^n \mathsf{E}[(\xi_j, y)^2] : \|y\| \le 1\}$ and C_1 is a constant depending on Q only.

Proof. See Theorem 5 of Einmahl (1993).

Lemma 3.2 *Define $\Delta_n := \|\overline{S}'_{nn} - S_n\|$. Let $r > 1$, $a > -d/2$ and $p > 0$. Suppose that the conditions (1.3) and (1.4) are satisfied. And let $\{f_n\}$ be a sequence of non-negative numbers satisfying (1.2). Then for any $\lambda > 0$, there exists a constant $K = K(r, a, p, \lambda)$ such that*

$$\sum_{n=1}^{\infty} n^{r-2} f_n I_n \le K \mathsf{E}\Big[\|X\|^{2r} (\log \|X\|)^{a-r}\Big] < \infty, \tag{3.3}$$

where

$$I_n = \mathsf{P}\Big\{\Delta_n \ge \sqrt{n}/(\log n)^2, \|\overline{S}'_{nn}\| \ge \lambda\phi(n)\Big\}.$$

Proof. For $n \ge 1$, let $\beta_n = n\mathsf{E}[\|X\| I\{\|X\| > \sqrt{n}/(\log n)^p\}]$. Then $\|\mathsf{E}\sum_{i=1}^j X'_{ni}\| \le \beta_n$, $1 \le j \le n$. Setting

$$\mathcal{L} = \{n : \beta_n \le \frac{1}{2}\sqrt{n}/(\log n)^2\},$$

then we have

$$\{\Delta_n \ge \sqrt{n}/(\log n)^2\} \subset \bigcup_{j=1}^n \{X_j \ne X'_{nj}\}, \quad n \in \mathcal{L}.$$

So for $n \in \mathcal{L}$,

$$I_n \le \sum_{j=1}^n \mathsf{P}\Big\{X_j \ne X'_{nj}, \|\overline{S}'_{nn}\| \ge \lambda\phi(n)\Big\}.$$

Observe that $X'_{nj} = 0$ whenever $X_j \ne X'_{nj}$, for $1 \le j \le n$, so that for any $\lambda > 0$, there exists $n_0 = n_0(\lambda)$ such that for $n \ge n_0$ and all $1 \le j \le n$, we have

$$\begin{aligned}
& \mathsf{P}\Big\{X_j \ne X'_{nj}, \|\overline{S}'_{nn}\| \ge \lambda\phi(n)\Big\} \\
= \ & \mathsf{P}\Big\{X_j \ne X'_{nj}, \|\sum_{i=1}^{j-1} \overline{X}'_{ni} + \sum_{i=j+1}^{n} \overline{X}'_{ni}\| \ge \lambda\phi(n)\Big\} \\
= \ & \mathsf{P}\Big\{X_j \ne X'_{nj}\Big\}\mathsf{P}\Big\{\|\sum_{i=1}^{j-1} \overline{X}'_{ni} + \sum_{i=j+1}^{n} \overline{X}'_{ni}\| \ge \lambda\phi(n)\Big\} \\
\le \ & \mathsf{P}\Big\{X_j \ne X'_{nj}\Big\}\mathsf{P}\Big\{\|\overline{S}'_{nn}\| \ge \lambda\phi(n) - 2\sqrt{n}/(\log n)^p\Big\} \\
\le \ & \mathsf{P}\Big\{\|X\| > \sqrt{n}/(\log n)^p\Big\}\mathsf{P}\Big\{\|\overline{S}'_{nn}\| \ge \lambda\phi(n)/2\Big\}.
\end{aligned}$$

By Lemma 3.1, (1.3) and (3.1), for any $Q > 2r$ there exist constants $C_1 = C_1(Q, \lambda) > 0$ and $\eta = \eta(\lambda) > 0$ such that for n large enough,

$$\begin{aligned}
& \mathsf{P}\Big\{\|\overline{S}'_{nn}\| \ge \frac{\lambda}{2}\phi(n)\Big\} \\
\le \ & \exp\Big\{-\frac{(\lambda\phi(n)/2 - 18Q^2\mathsf{E}\|\overline{S}'_{nn}\|)^2}{144\Lambda_n}\Big\} + CC_1\frac{n\mathsf{E}\big[\|X\|^Q I\{\|X\| \le \sqrt{n}/(\log n)^p\}\big]}{\big(\lambda\phi(n)/2 - 18Q^2\mathsf{E}\|\overline{S}'_{nn}\|\big)^Q} \\
& \qquad \text{(where } \Lambda'_n := \sup\big\{\textstyle\sum_{j=1}^n \mathsf{E}\big[(\overline{X}'_{nj}, y)^2\big] : \|y\| \le 1\big\} \le Cn, \text{ by (1.3))} \\
\le \ & n^{-\eta} + C_1C\big(\phi(n)\big)^{-Q} \cdot n\mathsf{E}\Big[\|X\|^Q I\{\|X\| \le \sqrt{n}/(\log n)^p\}\Big] \\
\le \ & n^{-\eta} + Cn^{1-r}(\log n)^{-\frac{Q}{2} - p(Q-2r) - a - r}\mathsf{E}\left[\|X\|^{2r}(\log\|X\|)^{a-r} I\{\|X\| \le \sqrt{n}/(\log n)^p\}\right] \\
\le \ & n^{-\eta} + Cn^{1-r}(\log n)^{-\frac{Q}{2} - p(Q-2r) - a - r} \le Cn^{-\nu},
\end{aligned} \tag{3.4}$$

where $0 < \nu < \min(\eta, r-1)$. So, by (1.2) and (1.4), we get

$$\begin{aligned}
\sum_{n \in \mathcal{L}} n^{r-2} f_n I_n &\le C\sum_{n=1}^\infty n^{r-2} f_n \cdot n\mathsf{P}\Big\{\|X\| > \frac{\sqrt{n}}{(\log n)^p}\Big\} \cdot n^{-\nu} \\
&\le C\sum_{n=1}^\infty n^{r-1-\nu} f_n \sum_{j=n}^\infty \mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}\mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}\sum_{n=1}^{j}n^{r-1-\nu}f_n\\
&\le C\sum_{j=1}^{\infty}\mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}\sum_{n=1}^{j}n^{r-1-\nu}(\log n)^a\\
&\qquad\text{(by Lemma 2.3)}\\
&\le C\sum_{j=1}^{\infty}\mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}j^{r-\nu}(\log j)^a\\
&\le C\mathsf{E}\Big[\|X\|^{2(r-\nu)}(\log\|X\|)^{a+2p(r-\nu)}\Big]\\
&\le C\mathsf{E}\Big[\|X\|^{2r}(\log\|X\|)^{a-r}\Big]<\infty.
\end{aligned}
$$

If $n\notin\mathcal{L}$, then we have

$$
I_n\le\mathsf{P}\Big\{\|\overline{S}'_{nn}\|\ge\lambda\phi(n)\Big\}\le Cn^{-\nu},
$$

by (3.4). It follows that

$$
\begin{aligned}
&\sum_{n\notin\mathcal{L}}n^{r-2}f_nI_n\le C\sum_{n\notin\mathcal{L}}n^{r-2-\nu}f_n\\
\le\;& C\sum_{n\notin\mathcal{L}}n^{r-3/2-\nu}f_n(\log n)^2\mathsf{E}\Big[\|X\|I\{\|X\|>\sqrt{n}/(\log n)^p\}\Big]\\
\le\;& C\sum_{n=1}^{\infty}n^{r-3/2-\nu}f_n(\log n)^2\sum_{j=n}^{\infty}\mathsf{E}\Big[\|X\|I\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big]\\
\le\;& C\sum_{j=1}^{\infty}\mathsf{E}\Big[\|X\|I\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big]\sum_{n=1}^{j}n^{r-3/2-\nu}f_n(\log n)^2\\
\le\;& C\sum_{j=1}^{\infty}\mathsf{E}\Big[\|X\|I\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big]\sum_{n=1}^{j}n^{r-3/2-\nu}(\log n)^{2+a}\\
&\qquad\text{(by Lemma 2.3)}\\
\le\;& C\sum_{j=1}^{\infty}\mathsf{E}\Big[\|X\|I\{\frac{\sqrt{j}}{(\log j)^p}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big]j^{r-1/2-\nu}(\log j)^{2+a}\\
\le\;& C\mathsf{E}\Big[\|X\|^{2r-2\nu}(\log\|X\|)^{2+a+2p(r-\nu-1/2)}\Big]\le C\mathsf{E}\Big[\|X\|^{2r}(\log\|X\|)^{a-r}\Big]<\infty.
\end{aligned}
$$

Now (3.3) is proved.

Lemma 3.3 *Let $r>1$, $a>-d/2$ and $p>0$. Suppose that the conditions (1.3) and (1.4) are satisfied and $\{f_n\}$ is a sequence of non-negative numbers satisfying (1.2). Then for any $\lambda>0$ there exists a constant $K=K(r,a,p,\lambda)$ such that*

$$
\sum_{n=1}^{\infty}n^{r-2}f_nII_n\le K\mathsf{E}\Big[\|X\|^{2r}(\log\|X\|)^{a-r}\Big]<\infty,\tag{3.5}
$$

where

$$
II_n=\mathsf{P}\Big\{\Delta_n\ge\sqrt{n}/(\log n)^2,\|S_n\|\ge\lambda\phi(n)\Big\}
$$

and Δ_n is as in Lemma 3.2.

Proof. Obviously,

$$\begin{aligned} II_n &\le \mathsf{P}\Big\{\Delta_n \ge \sqrt{n}/(\log n)^2, \|\overline{S}'_{nn}\| \ge \frac{\lambda}{3}\phi(n)\Big\} \\ &\quad + \mathsf{P}\Big\{\|\overline{S}''_{nn}\| \ge \frac{\lambda}{3}\phi(n)\Big\} + \mathsf{P}\Big\{\|\overline{S}'''_{nn}\| \ge \frac{\lambda}{3}\phi(n)\Big\}. \end{aligned}$$

Observe that

$$\begin{aligned} &\|\mathsf{E}S'''_{nn}\| \le Cn\mathsf{E}\big[\|X\|I\{\|X\| > \phi(n)\}\big] \le Cn\frac{\log n}{\phi(n)}\mathsf{E}\big[\frac{\|X\|^2}{\log\|X\|}I\{\|X\| > \phi(n)\}\big] \\ \le\ & C\phi(n)\mathsf{E}\big[\frac{\|X\|^2}{\log\|X\|}I\{\|X\| > \phi(n)\}\big] = o(\phi(n)), \end{aligned}$$

by (1.4). So we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|\overline{S}'''_{nn}\| \ge \frac{\lambda}{3}\phi(n)\Big\} \le C\sum_{n=1}^{\infty} n^{r-2} f_n \sum_{j=1}^{n} \mathsf{P}\Big\{X'''_j \ne 0\Big\} \\ \le\ & C\sum_{n=1}^{\infty} n^{r-1} f_n \mathsf{P}\Big\{X'''_j \ne 0\Big\} \le C\sum_{n=1}^{\infty} n^{r-1} f_n \mathsf{P}\big\{\|X\| > \phi(n)\big\} \\ \le\ & C\sum_{n=1}^{\infty} n^{r-1} f_n \sum_{j=n}^{\infty} \mathsf{P}\big\{\phi(j) < \|X\| \le \phi(j+1)\big\} \\ \le\ & C\sum_{j=1}^{\infty} \mathsf{P}\big\{\phi(j) < \|X\| \le \phi(j+1)\big\} \sum_{n=1}^{j} n^{r-1} f_n \\ \le\ & C\sum_{j=1}^{\infty} \mathsf{P}\big\{\phi(j) < \|X\| \le \phi(j+1)\big\} \sum_{n=1}^{j} n^{r-1}(\log n)^a \qquad \text{(by Lemma 2.3)} \\ \le\ & C\sum_{j=1}^{\infty} \mathsf{P}\big\{\phi(j) < \|X\| \le \phi(j+1)\big\} j^r(\log j)^a \\ \le\ & K\mathsf{E}\Big[\|X\|^{2r}(\log\|X\|)^{a-r}\Big] < \infty. \end{aligned} \tag{3.6}$$

Recall (1.3), we get that

$$\begin{aligned} &\overline{\Lambda}''_n := \sup\{\sum_{j=1}^{n} \mathsf{E}[(\overline{X}''_{nj}, y)^2] : \|y\| \le 1\} \\ \le\ & Cn \cdot \sup\big\{\mathsf{E}\big[\big(XI\{\sqrt{n}/(\log n)^p < \|X\| \le \phi(n)\} \\ &\quad -\mathsf{E}\big(XI\{\sqrt{n}/(\log n)^p < \|X\| \le \phi(n)\}\big), y\big)^2\big] : \|y\| \le 1\big\} \\ =\ & o(n). \end{aligned}$$

By Lemma 3.1, (1.2), Lemma 2.6, (3.1) and (1.4), we have for any $Q > 2r$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|\overline{S}''_{nn}\| \ge \frac{\lambda}{3}\phi(n)\Big\} \\ \le\ & C\sum_{n=1}^{\infty} n^{r-2} f_n \Big(\exp\big\{-\frac{(\frac{\lambda}{6}\phi(n))^2}{144\cdot o(n)}\big\} + \sum_{j=1}^{n}\frac{\mathsf{E}\|\overline{X}''_j\|^Q}{\big(\frac{\lambda}{6}\phi(n)\big)^Q}\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \quad K + C\sum_{n=1}^{\infty} n^{r-1} f_n \phi^{-Q}(n) \sum_{j=1}^{n} \mathsf{E}\big[\|X\|^Q I\{\phi(j-1) < \|X\| \le \phi(j)\}\big] \\
&\le \quad K + C\sum_{j=1}^{\infty} \mathsf{E}\big[\|X\|^Q I\{\phi(j-1) < \|X\| \le \phi(j)\}\big] \sum_{n=j}^{\infty} n^{r-1} f_n \phi^{-Q}(n) \\
&\le \quad K + C\sum_{j=1}^{\infty} \mathsf{E}\big[\|X\|^Q I\{\phi(j-1) < \|X\| \le \phi(j)\}\big] \\
&\qquad\qquad \cdot \Big[\sum_{n=j}^{\infty} n^{r-1}\phi^{-Q}(n)(\log n)^a + \sum_{n=1}^{j-1} (\log n)^a j^{r-1}\phi^{-Q}(j)\Big] \\
&\le \quad K + C\sum_{j=1}^{\infty} \mathsf{E}\big[\|X\|^Q I\{\phi(j-1) < \|X\| \le \phi(j)\}\big] j^{r-Q/2}(\log j)^{a-Q/2} \\
&\le \quad K + C\mathsf{E}\Big[\|X\|^{2r}(\log\|X\|)^{a-r}\Big] < \infty. \qquad (3.7)
\end{aligned}
$$

Finally, by noticing Lemma 3.2, we complete the proof of Lemma 3.3.

Lemma 3.4 *Let $1 < r < 3/2$, $a > -d/2$ and $p \ge \frac{6+r}{3-2r}$. Suppose that the condition (1.4) is satisfied and $\{f_n\}$ is a sequence of non-negative numbers satisfying (1.2). Then we have*

$$
\sum_{n=1}^{\infty} n^{r-2} f_n p_n < \infty, \qquad (3.8)
$$

where $p_n = \big(\frac{\sqrt{n}}{(\log n)^2}\big)^{-3} \sum_{j=1}^{n} \mathsf{E}[\|\overline{X}'_j\|]^3$.

Proof. Observe that $p_n \le C\big(\frac{\sqrt{n}}{(\log n)^2}\big)^{-3} \sum_{j=1}^{n} \mathsf{E}[\|X'_j\|]^3$. So by (1.2) and Lemma 2.6, for $1 < r < 3/2$, we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} f_n p_n \\
&\le \quad C\sum_{n=1}^{\infty} n^{r-2} f_n n^{-3/2}(\log n)^6 \cdot n\mathsf{E}\big[\|X\|^3 I\big\{\|X\| \le \frac{\sqrt{n}}{(\log n)^p}\big\}\big] \\
&\le \quad C\sum_{n=1}^{\infty} n^{r-5/2}(\log n)^6 f_n \sum_{j=1}^{n} \mathsf{E}\Big[\|X\|^3 I\big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\big\}\Big] \\
&\le \quad C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\|^3 I\big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\big\}\Big] \sum_{n=j}^{\infty} n^{r-5/2}(\log n)^6 f_n \\
&\le \quad C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\|^3 I\big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\big\}\Big] \\
&\qquad\qquad \cdot \Big[\sum_{n=j}^{\infty} n^{r-5/2}(\log n)^{6+a} + \sum_{n=1}^{j-1} (\log n)^a j^{r-5/2}(\log j)^6\Big] \\
&\le \quad C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\|^3 I\big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\big\}\Big] j^{r-3/2}(\log j)^{6+a} \\
&\le \quad C\mathsf{E}\big[\|X\|^{2r}(\log\|X\|)^{a-r}\big] < \infty,
\end{aligned}
$$

whenever $p \ge \frac{6+r}{3-2r}$. So (3.8) is proved.

Lemma 3.5 *(Einmahl, 1991) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent H-valued r.v.'s with $\mathsf{E}\xi_j = 0$, $\mathsf{E}[\|\xi_j\|^3] < \infty$ and let $Y_1, Y_2, \cdots, Y_n$ be independent Gaussian mean zero r.v.'s with $\mathsf{Cov}(\xi_j) = \mathsf{Cov}(Y_j)$, $j = 1, 2, \cdots, n$. Then we have for any $s, t > 0$,*

$$\mathsf{P}\{\|\sum_{j=1}^n \xi_j\| \geq s\} \leq \mathsf{P}\{\|\sum_{j=1}^n Y_j\| \geq s - t\} + C_2 t^{-3} \sum_{j=1}^n \mathsf{E}[\|\xi_j\|^3] \tag{3.9}$$

and

$$\mathsf{P}\{\|\sum_{j=1}^n \xi_j\| \geq s\} \geq \mathsf{P}\{\|\sum_{j=1}^n Y_j\| \geq s + t\} - C_2 t^{-3} \sum_{j=1}^n \mathsf{E}[\|\xi_j\|^3], \tag{3.10}$$

where C_2 is a universal constant.

Now we turn to prove the theorems. Let $\{Y'_{nj}\}$ be a sequence of independent H-valued Gaussian mean zero random variables with $\Sigma_n := \mathsf{Cov}(Y'_{nj}) = \mathsf{Cov}(\overline{X}'_{nj})$, $1 \leq j \leq n$. Write $T'_n := \sum_{j=1}^n Y'_{nj}$, for $n \geq 1$. Recall $\Sigma := \mathsf{Cov}(X)$.

Proof of Theorem 1.1. Take $p \geq \frac{6+r}{3-2r}$, for $1 < r < 3/2$. Applying the inequality of Anderson (1955), we get for any $x \in \mathbb{R}$:

$$\mathsf{P}\{\|T'_n\| \leq x\} \geq \mathsf{P}\{\|Y\| \leq x/\sqrt{n}\}, \qquad n \in \mathbb{N}, \tag{3.11}$$

where Y is a Gaussian r.v. with mean zero and covariance operator Σ.

Let $0 < \delta < \sqrt{r-1}/4$. For $n \geq 1$, Eq. (3.9) with $s = \varepsilon\phi(n) - \frac{\sqrt{n}}{(\log n)^2} > 0$ and $t = \frac{\sqrt{n}}{(\log n)^2} > 0$ and Eq. (3.11) yield

$$\begin{aligned}
& \mathsf{P}\{\|S_n\| \geq \varepsilon\phi(n)\} \\
= \;& \mathsf{P}\{\|S_n\| \geq \varepsilon\phi(n), \Delta_n < \frac{\sqrt{n}}{(\log n)^2}\} + \mathsf{P}\{\|S_n\| \geq \varepsilon\phi(n), \Delta_n \geq \frac{\sqrt{n}}{(\log n)^2}\} \\
\leq \;& \mathsf{P}\{\|S_n\| \geq \varepsilon\phi(n), \Delta_n < \frac{\sqrt{n}}{(\log n)^2}\} + \mathsf{P}\{\|S_n\| \geq \frac{3\sqrt{r-1}}{4}\phi(n), \Delta_n \geq \frac{\sqrt{n}}{(\log n)^2}\} \\
\leq \;& \mathsf{P}\{\|\overline{S}'_{nn}\| \geq \varepsilon\phi(n) - \frac{\sqrt{n}}{(\log n)^2}\} + II_n \\
\leq \;& \mathsf{P}\{\|T'_n\| \geq \varepsilon\phi(n) - \frac{2\sqrt{n}}{(\log n)^2}\} + C_2 p_n + II_n \\
\leq \;& \mathsf{P}\{\|Y\| \geq \varepsilon\sqrt{2\log n} - \frac{2}{(\log n)^2}\} + C_2 p_n + II_n,
\end{aligned} \tag{3.12}$$

for all $\varepsilon \in \left(\sqrt{r-1} - \delta, \sqrt{r-1} + \delta\right)$, where II_n is defined in Lemma 3.3 with $\lambda = \frac{3\sqrt{r-1}}{4}$ and p_n is defined in Lemma 3.4. If we let

$$a'_n(\varepsilon) = a_n(\varepsilon) - \sqrt{2}/(\log n)^{5/2}.$$

Then $a'_n(\varepsilon)$ satisfies the condition (1.1) and

$$\mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n(\varepsilon))\phi(n)\} \leq \mathsf{P}\{\|Y\| \geq (\varepsilon + a'_n(\varepsilon))\sqrt{2\log n}\} + C_2 p_n + II_n,$$

for all $\varepsilon \in (\sqrt{r-1} - \delta/2, \sqrt{r-1} + \delta/2)$ by (3.12). So, by Proposition 2.1, Lemma 3.3 and Lemma 3.4, it follows that

$$\begin{aligned}
&\limsup_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|S_n\| \geq \phi(n)\big(\varepsilon + a_n(\varepsilon)\big)\Big\} \\
&\qquad \leq \Gamma^{-1}(d/2) K(\Sigma)(r-1)^{\frac{d-2}{2}} \Gamma(a + d/2) \exp\{-2\tau\sqrt{r-1}\}. \qquad (3.13)
\end{aligned}$$

Now we consider the lower bound of (1.6). Firstly, we consider the finite dimension case, i.e., $d' < \infty$. Notice that Σ^{-1} exists and $\Sigma_n \to \Sigma$, as $n \to \infty$. So, we can also assume that Σ_n^{-1} exists for all $n \geq 1$. Using (3.10) instead of (3.9), similar to (3.12) we have

$$\begin{aligned}
&\mathsf{P}\{\|S_n\| \geq \varepsilon\phi(n)\} \geq \mathsf{P}\{\|S_n\| \geq \varepsilon\phi(n), \Delta_n < \frac{\sqrt{n}}{(\log n)^2}\} \\
\geq\ & \mathsf{P}\{\|\overline{S}'_{nn}\| \geq \varepsilon\phi(n) + \frac{\sqrt{n}}{(\log n)^2}, \Delta_n < \frac{\sqrt{n}}{(\log n)^2}\} \\
\geq\ & \mathsf{P}\{\|\overline{S}'_{nn}\| \geq \varepsilon\phi(n) + \frac{\sqrt{n}}{(\log n)^2}\} - \mathsf{P}\{\|\overline{S}'_{nn}\| \geq \varepsilon\phi(n) + \frac{\sqrt{n}}{(\log n)^2}, \Delta_n \geq \frac{\sqrt{n}}{(\log n)^2}\} \\
\geq\ & \mathsf{P}\{\|\overline{S}'_{nn}\| \geq \varepsilon\phi(n) + \frac{\sqrt{n}}{(\log n)^2}\} - \mathsf{P}\{\|\overline{S}'_{nn}\| \geq \frac{3\sqrt{r-1}}{4}\phi(n), \Delta_n \geq \frac{\sqrt{n}}{(\log n)^2}\} \\
\geq\ & \mathsf{P}\{\|T'_n\| \geq \varepsilon\phi(n) + \frac{2\sqrt{n}}{(\log n)^2}\} - C_2 p_n - I_n, \qquad (3.14)
\end{aligned}$$

for all $\varepsilon \in \big(\sqrt{r-1} - \delta, \sqrt{r-1} + \delta\big)$ and $n \geq 1$, where I_n is defined in Lemma 3.2 with $\lambda = \frac{3\sqrt{r-1}}{4}$. Write $\Xi_n = \Sigma_n^{1/2}\Sigma^{-1/2}$ and

$$\gamma_n := \|\Xi_n^{-1}\| = \sup_{y \neq 0} \|\Xi_n^{-1} y\|/\|y\|. \qquad (3.15)$$

Then

$$\|Y\| = \|\Xi_n^{-1}\Xi_n Y\| \leq \|\Xi_n^{-1}\| \cdot \|\Xi_n Y\| = \gamma_n \|\Xi_n Y\|.$$

We conclude that for any $x > 0$,

$$\begin{aligned}
&\mathsf{P}\{\|T'_n\| \geq x\sqrt{n}\} = \mathsf{P}\{\|Y'_{ni}\| \geq x\} \\
=\ & \mathsf{P}\{\|\Sigma_n^{1/2}\Sigma^{-1/2} Y\| \geq x\} \geq \mathsf{P}\{\|Y\| \geq x\gamma_n\}. \qquad (3.16)
\end{aligned}$$

Also,

$$\begin{aligned}
&|\gamma_n^2 - 1| = \Big|\|\Sigma^{1/2}\Sigma_n^{-1/2}\|^2 - 1\Big| = \Big|\|\Sigma_n^{-1/2}\Sigma\Sigma_n^{-1/2}\| - \|\Sigma_n^{-1/2}\Sigma_n\Sigma_n^{-1/2}\|\Big| \\
&\leq \|\Sigma_n^{-1/2}(\Sigma - \Sigma_n)\Sigma_n^{-1/2}\| \leq \|\Sigma_n^{-1/2}\|^2 \cdot \|\Sigma_n - \Sigma\| \\
&= \|\Sigma_n^{-1}\| \cdot \|\Sigma_n - \Sigma\| \leq C\|\Sigma_n - \Sigma\| = o\left(1/\log n\right),
\end{aligned}$$

by (1.5). It follows that

$$\gamma_n = 1 + o\left(1/\log n\right). \tag{3.17}$$

If we let

$$a''_n(\varepsilon) = \Big(\varepsilon + a_n(\varepsilon) + \sqrt{2}/(\log n)^{5/2}\Big)\gamma_n - \varepsilon.$$

Then by (3.14) and (3.16),

$$\begin{aligned}
& \mathsf{P}\{\|Y\| \geq (\varepsilon + a''_n(\varepsilon))\sqrt{2\log n}\} - C_2 p_n - I_n \\
= \;& \mathsf{P}\{\|Y\| \geq \gamma_n[(\varepsilon + a_n(\varepsilon))\sqrt{2\log n} + \frac{2}{(\log n)^2}]\} - C_2 p_n - I_n \\
\leq \;& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n(\varepsilon))\phi(n)\},
\end{aligned}$$

for all $\varepsilon \in (\sqrt{r-1} - \delta/2, \sqrt{r-1} + \delta/2)$ and n large enough, by the condition $a_n(\varepsilon) \to 0$ again. It follows from Lemma 3.2, Lemma 3.4 and Proposition 2.1 that

$$\begin{aligned}
& \liminf_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|S_n\| \geq \phi(n)\big(\varepsilon + a_n(\varepsilon)\big)\Big\} \\
& \qquad \geq \Gamma^{-1}(d/2) K(\Sigma)(r-1)^{\frac{d-2}{2}} \Gamma(a + d/2) \exp\{-2\tau\sqrt{r-1}\}.
\end{aligned}$$

Hence, in the finite dimension case, the lower bound of (1.6) is proved. Now assume $d' = \infty$. For any $d'' \geq d$, let $Q : \mathbf{H} \to \mathbf{H}$ be the projection onto the d''-dimensional eigenspace of σ_i^2, $i = 1, \ldots, d''$, i.e., $Q(y) = \sum_{i=1}^{d''} (y, e_i) e_i$, $y \in \mathbf{H}$. Since $\|Q(y)\| \leq \|y\|$, $y \in \mathbf{H}$, from the special case proved above, it follows that

$$\begin{aligned}
& \liminf_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|S_n\| \geq \phi(n)\big(\varepsilon + a_n(\varepsilon)\big)\Big\} \\
\geq \;& \liminf_{\varepsilon \searrow \sqrt{r-1}} [\varepsilon^2 - (r-1)]^{a+d/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{\|Q(S_n)\| \geq \phi(n)\big(\varepsilon + a_n(\varepsilon)\big)\Big\} \\
\geq \;& \Gamma^{-1}(d/2) K_{d''}(\Sigma)(r-1)^{\frac{d-2}{2}} \Gamma(a + d/2) \exp\{-2\tau\sqrt{r-1}\},
\end{aligned}$$

where $K_{d''}(\Sigma) = \prod_{i=d+1}^{d''} (1 - \sigma_i^2/\sigma^2)^{-1/2}$. Letting $d'' \to \infty$. we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Take $p = 2$. Write

$$q_n = \mathsf{P}\{\Delta_n > \sqrt{n}/(\log n)^2\}, \quad p_n = \Big(\frac{\sqrt{n}}{(\log n)^2}\Big)^{-3} \sum_{j=1}^{n} \mathsf{E}\big[\|\overline{X}'_j\|\big]^3.$$

Following the lines in the proof of (3.3), by (1.8) and Lemma 2.3, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{-1} f_n q_n = \sum_{n \in \mathcal{L}} n^{-1} f_n q_n + \sum_{n \notin \mathcal{L}} n^{-1} f_n q_n \\
\leq \;& \sum_{n \in \mathcal{L}} n^{-1} f_n \sum_{j=1}^{n} \mathsf{P}\{X_j \neq X'_{nj}\} + \sum_{n \notin \mathcal{L}} n^{-1} f_n \frac{(\log n)^2}{\sqrt{n}} \cdot 2\beta_n \\
\leq \;& \sum_{n \in \mathcal{L}} f_n \mathsf{P}\{\|X\| > \sqrt{n}/(\log n)^p\} + 2\sum_{n \notin \mathcal{L}} n^{-1/2} (\log n)^2 f_n \mathsf{E}\Big[\|X\| I\{\|X\| > \sqrt{n}/(\log n)^p\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} f_n \sum_{j=n}^{\infty} \mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\} \\
&\qquad +2\sum_{n=1}^{\infty} n^{-1/2}(\log n)^2 f_n \sum_{j=n}^{\infty} \mathsf{E}\Big[\|X\| I\Big\{\frac{\sqrt{j}}{(\log n)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}\Big] \\
&= \sum_{j=1}^{\infty} \mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\} \sum_{n=1}^{j} f_n \\
&\qquad +2\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\| I\Big\{\frac{\sqrt{j}}{(\log n)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}\Big] \sum_{n=1}^{j} n^{-1/2}(\log n)^2 f_n \\
&\le C\sum_{j=1}^{\infty} \mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\} \sum_{n=1}^{j} (\log n)^a \\
&\qquad +C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\| I\Big\{\frac{\sqrt{j}}{(\log n)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}\Big] \sum_{n=1}^{j} n^{-1/2}(\log n)^{2+a} \\
&\le C\sum_{j=1}^{\infty} \mathsf{P}\Big\{\frac{\sqrt{j}}{(\log j)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\} j(\log j)^a \\
&\qquad +C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\| I\Big\{\frac{\sqrt{j}}{(\log n)^p} < \|X\| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big\}\Big] j^{1/2}(\log j)^{2+a} \\
&\le C\mathsf{E}\big[\|X\|^2(\log\|X\|)^{2p+a}\big] + C\mathsf{E}\big[\|X\|^2(\log\|X\|)^{p+a+2}\big] \\
&\le C\mathsf{E}\big[\|X\|^2(\log\|X\|)^{a+4}\big] < \infty, \qquad (3.18)
\end{aligned}
$$

by recalling $p = 2$. Also, using Lemma 2.6, we can get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} f_n p_n \\
&\le C\sum_{n=1}^{\infty} n^{-3/2}(\log n)^6 f_n \sum_{j=1}^{n} \mathsf{E}\Big[\|X\|^3 I\Big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\Big\}\Big] \\
&= C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\|^3 I\Big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\Big\}\Big] \sum_{n=j}^{\infty} n^{-3/2}(\log n)^6 f_n \\
&\le C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\|^3 I\Big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\Big\}\Big] \\
&\qquad \cdot\Big\{\sum_{n=j}^{\infty} n^{-3/2}(\log n)^{6+a} + \sum_{n=1}^{j-1}(\log n)^a j^{-3/2}(\log j)^6 \\
&\le C\sum_{j=1}^{\infty} \mathsf{E}\Big[\|X\|^3 I\Big\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < \|X\| \le \frac{\sqrt{j}}{(\log j)^p}\Big\}\Big] \cdot j^{-1/2}(\log j)^{6+a} \\
&\le C\mathsf{E}\big[\|X\|^2(\log\|X\|)^{6+a-p} = C\mathsf{E}\big[\|X\|^2(\log\|X\|)^{a+4} < \infty. \qquad (3.19)
\end{aligned}
$$

Without losing generality, we can assume that $|a_n| \le 1/\log n$. Similar to the proof of Theorem 1.1, we have, for n large enough and all $\varepsilon > 0$,

$$
\begin{aligned}
& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n)\} \\
= \;& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n), \quad \Delta_n \leq \frac{\sqrt{n}}{(\log n)^2}\} + \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n), \quad \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\} \\
\leq \;& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n), \quad \Delta_n \leq \frac{\sqrt{n}}{(\log n)^2}\} + q_n \\
\leq \;& \mathsf{P}\{\|\overline{S}'_{nn}\| \geq (\varepsilon + a_n)\phi(n) - \frac{\sqrt{n}}{(\log n)^2}\} + q_n \\
\leq \;& \mathsf{P}\{\|T'_n\| \geq (\varepsilon + a_n)\phi(n) - \frac{2\sqrt{n}}{(\log n)^2}\} + C_2 p_n + q_n \\
\leq \;& \mathsf{P}\{\|Y\| \geq \big(\varepsilon + a_n - \sqrt{2}/(\log n)^{5/2}\big)\sqrt{2\log n}\} + C_2 p_n + q_n \\
\leq \;& \mathsf{P}\{\|Y\| \geq \big(\varepsilon - 2/\log n\big)\sqrt{2\log n}\} + C_2 p_n + q_n, \qquad (3.20)
\end{aligned}
$$

where $\Delta_n = \|\overline{S}'_{nn} - S_n\|$. Combing (3.18)-(3.20) and applying Proposition 2.2 yield the upper bound of (1.9). For the lower bound, it also suffices to consider the case of the finite dimension case, i.e., $d' < \infty$. Notice that for n large enough,

$$
\begin{aligned}
& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n)\} \\
\geq \;& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n), \quad \Delta_n \leq \frac{\sqrt{n}}{(\log n)^2}\} \\
\geq \;& \mathsf{P}\{\|\overline{S}'_{nn}\| \geq (\varepsilon + a_n)\phi(n) + \frac{\sqrt{n}}{(\log n)^2}, \quad \Delta_n \leq \frac{\sqrt{n}}{(\log n)^2}\} \\
\geq \;& \mathsf{P}\{\|\overline{S}'_{nn}\| \geq (\varepsilon + a_n)\phi(n) + \frac{\sqrt{n}}{(\log n)^2}\} - q_n \\
\geq \;& \mathsf{P}\{\|T'_n\| \geq (\varepsilon + a_n)\phi(n) + \frac{2\sqrt{n}}{(\log n)^2}\} - C_2 p_n - q_n \\
\geq \;& \mathsf{P}\{\|T'_n\| \geq \big(\varepsilon + 2/\log n\big)\phi(n)\} - C_2 p_n - q_n \\
\geq \;& \mathsf{P}\{\|Y\| \geq \gamma_n\big(\varepsilon + 2/\log n\big)\sqrt{2\log n}\} - C_2 p_n - q_n, \qquad (3.21)
\end{aligned}
$$

by (3.10) and (3.16), where γ_n is defined in (3.15). Notice also that $\gamma_n \to 1$, as $n \to \infty$. Fix $1 < \theta < 2$. We conclude that for n large enough and all $\varepsilon > 0$,

$$
\begin{aligned}
& \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n)\} \\
\geq \;& \mathsf{P}\{\|Y\| \geq \big(\theta\varepsilon + 4/\log n\big)\sqrt{2\log n}\} - C_2 p_n - q_n, \qquad (3.22)
\end{aligned}
$$

by (3.21). Putting (3.18), (3.19), (3.22) together and applying Proposition 2.2 yield

$$
\begin{aligned}
& \liminf_{\varepsilon \searrow 0} \varepsilon^{2(a+1)} \sum_{n=1}^{\infty} n^{-1} f_n \mathsf{P}\{\|S_n\| \geq (\varepsilon + a_n)\phi(n)\} \\
\geq \;& \theta^{-2(a+1)}(a+1)^{-1}\mathsf{E}\big[\|Y\|\big]^{2(a+1)}.
\end{aligned}
$$

Letting $\theta \to 1$, we complete the proof of Theorem 1.2.

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